On the general expression of the conformally covariant energy-momentum tensor

1981 J. Phys. A: Math. Gen. 14 L125

(http://iopscience.iop.org/0305-4470/14/5/005)

J. Phys. A: Math. Gen. 14 (1981) L125–L127. Printed in Great Britain

LETTER TO THE EDITOR

On the general expression of the conformally covariant energy–momentum tensor

Bo-Wei Xu

Department of Physics, University of Colorado, Boulder, CO 80309 USA and Department of Modern Physics, Lanzhou University, Lanzhou, China

Received 23 February 1981

Abstract. We derive the general expression of the conformally covariant energy–momentum tensor.

In a previous work we proposed the conformally covariant energy–momentum tensors for the fields of spin 0, 1 and $\frac{1}{2}$, in terms of which other conformal currents can be expressed (Xu 1981). We shall now derive a general expression of the conformally covariant energy–momentum tensor by starting from the general form of the Lagrangian from which the field equations are conformally covariant.

The conformal group consists of the following transformations

(i) inhomogeneous Lorentz transformations

$$x'_\mu = \Lambda^\nu_\mu x_\nu + a_\mu \tag{1}$$

(ii) dilatation transformations

$$x'_\mu = \rho x_\mu \tag{2}$$

(iii) special conformal transformations

$$x'_\mu = (x_\mu + c_\mu x^2)/\Omega \qquad \Omega = 1 + 2cx + c^2x^2. \tag{3}$$

It has been shown that a set of fields $\phi_\alpha(x)$ belonging to a linear representation of the inhomogeneous Lorentz group behave under the conformal transformations as (Isham *et al* 1970)

$$\phi'_\alpha(x') = \left|\det \frac{\partial x'}{\partial x}\right|^{l/4} D^\beta_\alpha(\Lambda(x))\phi_\beta(x) \tag{4}$$

where

$$\Lambda_{\mu\nu}(x) = \left|\det \frac{\partial x'}{\partial x}\right|^{-1/4} \frac{\partial x'_\mu}{\partial x^\nu} \tag{5}$$

and l is the conformal weight of the field $\phi_\alpha(x)$. For the special conformal transformations, hence we have

$$\phi'_\alpha(x') = \Omega^{-l} D^\beta_\alpha(\Lambda(x))\phi_\beta(x) \tag{6}$$

where

$$D^\beta_\alpha(\Lambda(x)) = g^\beta_\alpha + (c^\mu x^\nu - x^\mu c^\nu) I^\beta_{\mu\nu\alpha}. \tag{7}$$

0305-4470/81/050125+03\$01.50

$I^{\beta}_{\mu\nu\alpha}$ denotes the usual spin matrices which satisfy the following relation

$$[I_{\mu\nu}, I_{\lambda\sigma}] = g_{\mu\sigma}I_{\nu\lambda} + g_{\nu\lambda}I_{\mu\sigma} - g_{\mu\lambda}I_{\nu\sigma} - g_{\nu\sigma}I_{\mu\lambda}. \tag{8}$$

Since the $\Lambda_{\mu\nu}(x)$ depends upon x_μ, the ordinary partial derivative $\partial_\mu\phi_\alpha(x)$ is not conformally covariant

$$\partial'_\mu\phi'_\alpha(x') = \Omega^{1-l}d^\lambda_\mu D^\beta_\alpha\, \partial_\lambda\phi_\beta + 2C^\lambda I^\beta_{\lambda\mu\alpha}\phi_\beta - 2lc_\mu\phi_\alpha \tag{9}$$

where

$$d^\lambda_\mu = g^\lambda_\mu + 2(c_\mu x^\lambda - x_\mu c^\lambda). \tag{10}$$

Now let us examine the transformations of the Lagrangian. Suppose $\mathscr{L}(\phi_\alpha, \partial_\mu\phi_\alpha)$ to be Poincaré invariant as well as dilatation covariant, then the Lagrangian under the special conformal transformations becomes as (Flato *et al* 1970)

$$\mathscr{L}'(\phi'_\alpha, \partial'_\mu\phi'_\alpha) = \Omega^4\mathscr{L}(\phi_\alpha, \partial_\mu\phi_\alpha) + 2c^\lambda R_\lambda \tag{11}$$

where

$$\begin{aligned} R_\lambda &= -\pi^{\sigma\alpha}(I^\beta_{\sigma\lambda\alpha}\phi_\beta + lg_{\lambda\sigma}\phi_\alpha) \\ \pi^{\sigma\alpha} &= \frac{\partial\mathscr{L}}{\partial\,\partial_\sigma\phi_\alpha}. \end{aligned} \tag{12}$$

We restrict ourselves to the case of $R_\lambda = 0$ or $R_\lambda = \partial_\lambda R$. R is some scalar function of fields $\phi_\alpha(x)$, and has the conformal weight $l_R = -2$. The field equations will thus be conformally covariant. The canonical energy–momentum tensor

$$T_{\mu\nu} = g_{\mu\nu}\mathscr{L} - \pi^\alpha_\mu\, \partial_\nu\phi_\alpha \tag{13}$$

transforms as

$$\begin{aligned} T'_{\mu\nu} = \Omega^4(g_{\mu\nu}\mathscr{L} - d^\sigma_\mu d^\lambda_\nu\pi^\beta_\sigma\, \partial_\lambda\phi_\beta) + 2c^\lambda\pi^\alpha_\mu I^\beta_{\nu\lambda\alpha}\phi_\beta \\ + 2c_\nu\pi^{\lambda\alpha}I^\beta_{\mu\lambda\alpha}\phi_\beta + 2g_{\mu\nu}c^\lambda R_\lambda - 2c_\mu R_\nu - 2c_\nu R_\mu \end{aligned} \tag{14}$$

so $T_{\mu\nu}$ is not conformally covariant. In order to cancel the nonconformally covariant terms in equation (14), we may add to $T_{\mu\nu}$ some extra Poincaré covariant terms such as

$$\partial^\lambda[(\pi^\alpha_\lambda I^\beta_{\mu\nu\alpha} + \pi^\alpha_\mu I^\beta_{\nu\lambda\alpha} + \pi^\alpha_\nu I^\beta_{\mu\lambda\alpha})\phi_\beta] \tag{15}$$

and

$$(g_{\mu\nu}\,\partial^2 - \partial_\mu\,\partial_\nu)R \tag{16}$$

and redefine the energy–momentum tensor $\Theta_{\mu\nu}$ as

$$\begin{aligned} \Theta_{\mu\nu} = T_{\mu\nu} + a\,\partial^\lambda[(\pi^\alpha_\lambda I^\beta_{\mu\nu\alpha} + \pi^\alpha_\mu I^\beta_{\nu\lambda\alpha} + \pi^\alpha_\nu I^\beta_{\mu\lambda\alpha})\phi_\beta] \\ + b(g_{\mu\nu}\,\partial^2 - \partial_\mu\,\partial_\nu)R. \end{aligned} \tag{17}$$

The constants a and b in equation (17) are determined in such a way that $\Theta_{\mu\nu}$ is conformally covariant. On using the above special conformal transformation properties, we can obtain the transformations of equations (15) and (16) respectively as

$$\begin{aligned}\partial'^{\lambda}[(\pi'^{\alpha}_{\lambda}I^{\beta}_{\mu\nu\alpha}+\pi'^{\alpha}_{\mu}I^{\beta}_{\nu\lambda\alpha}+\pi'^{\alpha}_{\nu}I^{\beta}_{\mu\lambda\alpha})\phi'_{\beta}]\\ =\Omega^{4}d^{\rho}_{\lambda}D^{\alpha}_{\gamma}D^{\delta}_{\beta}(d^{\sigma}_{\rho}I^{\beta}_{\mu\nu\alpha}+d^{\sigma}_{\mu}I^{\beta}_{\nu\rho\alpha}+d^{\sigma}_{\nu}I^{\beta}_{\mu\sigma\alpha})\,\partial^{\lambda}(\pi^{\gamma}_{\sigma}\phi_{\delta})\\ +4c^{\lambda}\pi^{\alpha}_{\mu}I^{\beta}_{\nu\lambda\alpha}\phi_{\beta}+4c_{\nu}\pi^{\lambda\alpha}I^{\beta}_{\mu\lambda\alpha}\phi_{\beta}\\ +2c^{\lambda}\,\partial^{\sigma}\left[\left(\frac{\partial R_{\lambda}}{\partial\,\partial^{\sigma}\phi_{\alpha}}I^{\beta}_{\mu\nu\alpha}+\frac{\partial R_{\lambda}}{\partial\,\partial^{\mu}\phi_{\alpha}}I^{\beta}_{\nu\sigma\alpha}+\frac{\partial R_{\lambda}}{\partial\,\partial^{\nu}\phi_{\alpha}}I^{\beta}_{\mu\sigma\alpha}\right)\phi_{\beta}\right]\end{aligned} \tag{18}$$

and

$$\begin{aligned}(g_{\mu\nu}\,\partial'^{2}-\partial'_{\mu}\,\partial'_{\nu})R'\\ =\Omega^{4}(g_{\mu\nu}\,\partial^{2}-d^{\rho}_{\mu}d^{\lambda}_{\nu}\,\partial_{\rho}\,\partial_{\lambda})R+6g_{\mu\nu}c^{\lambda}\,\partial_{\lambda}R-6c_{\mu}\,\partial_{\nu}R-6c_{\nu}\,\partial_{\mu}R.\end{aligned} \tag{19}$$

For the massless field of spin 0 ($R=-\frac{1}{2}\phi^{2}$) and that of spin $\frac{1}{2}$, 1 ($R_{\lambda}=0$), the last term in equation (18) is identical to zero. Therefore we can verify from equations (14) and (17)–(19) that

$$\Theta_{\mu\nu}=T_{\mu\nu}-\tfrac{1}{2}\partial^{\lambda}[(\pi^{\alpha}_{\lambda}I^{\beta}_{\mu\nu\alpha}+\pi^{\alpha}_{\mu}I^{\beta}_{\nu\lambda\alpha}+\pi^{\alpha}_{\nu}I^{\beta}_{\mu\lambda\alpha})\phi_{\beta}]-\tfrac{1}{3}(g_{\mu\nu}\,\partial^{2}-\partial_{\mu}\,\partial_{\nu})R \tag{20}$$

is conformally covariant, and has the properties

$$\partial^{\mu}\Theta_{\mu\nu}=0 \qquad \Theta_{\mu\nu}=\Theta_{\nu\mu} \qquad \Theta^{\mu}_{\mu}=0. \tag{21}$$

Equation (20) gives the general expression of the conformally covariant energy–momentum tensor.

I should like to thank Professor A O Barut for useful discussions.

References

Flato M, Simon J and Sternheimer D 1970 *Ann. Phys., NY* **61** 78
Isham C J, Salam A and Strathdee J 1970 *Phys. Lett.* **31B** 300
Xu B W 1981 *J. Phys. A: Math. Gen.* **14** L97